Worksheet- Set theory **Name:**
Part 2: 25/08/2024

Notations:
$\mathbb{N}$ is the set of all natural numbers, $\{1, 2, 3, 4, \cdots\}$,
$\mathbb{Z}$ is the set of all integers, $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
$\mathbb{R}$ is the set of all real numbers.

1. When is $A \times B = \varnothing$?

2. Let $X = \mathbb{R} = Y$. Visualize the cross-products $A \times B$ and $B \times A$ where $A \subseteq X$ and $B \subseteq Y$ are given below.

   (a) $A = [-1, 1]$, $B = [0, 1]$
   (b) $A = [-1, 1)$, $B = (0, 1]$
   (c) $A = [-1, 0) \cup (0, 1)$, $B = [0, 1]$
   (d) $A = \{1\}$, $B = \mathbb{R}$
   (e) $A = \mathbb{Z}$, $B = \mathbb{R}$

3. If A has m elements and B has n elements, how many elements does $A \times B$ have?

4. What's wrong with the following proof that for any sets $A,\ B,\ C$, and D, $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$?

   Proof. Suppose $(x, y) \in (A \cup C) \times (B \cup D)$. Then $x \in A \cup C$ and $y \in B \cup D$, so either $x \in A$ or $x \in C$, and either $y \in B$ or $y \in D$. We consider these cases separately.

   - Case 1 $x \in A$ and $y \in B$. Then $(x, y) \in A \times B$.
   - Case 2. $x \in C$ and $y \in D$. Then $(x, y) \in C \times D$.

   Thus, either $(x, y) \in A \times B$ or $(x, y) \in C \times D$, so $(x, y) \in (A \times B) \cup (C \times D)$. $\square$

5. Suppose A, B, C, and D are sets. Prove the following:

   (a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   (c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
   (d) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

6. Given a map $f : X \to Y$, think of a "natural" subset of $X \times Y$ associated with f.

7. Given a subset $A \times B \subset X \times Y$ and elements $(x_1, y_1), (x_2, y_2) \in A \times B$, can you think of a few other (possibly new) elements in $A \times B$?

8. Let $C = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$. Then $C \subseteq \mathbb{R}^2$ and it is not of the form $A \times B$ for any subsets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$.

9. Let C be the set of all cities, N the set of all countries, and let $L = \{(c, n) \in C \times N :$ the city c is in the country n $\}$. Is L a function from C to N?

10. Let P be the set of all people, and let $C = \{(p, q) \in P \times P :$ the person p is a parent of the person q$\}$. Is C a function from P to P?

11. Let P be the set of all people, and let $D = \{(p, x) \in P \times \mathcal{P}(P) :$ x = the set of all children of p$\}$. Is D a function from P to $\mathcal{P}$(P)?

12. Let W be the set of all words of English, and let A be the set of all letters of the alphabet. Let $f = \{(w, a) \in W \times A :$ the letter a occurs in the word w$\}$, and let $g = \{(w, a) \in W \times A :$ the letter a is the first letter of the word w$\}$. Is f a function from W to A? How about g?